

## Solutions, Tutorial 4

1.

a)

The maximum height of a Chapman layer is given by (Equation 3.5.8 in Fälthammar)

$$z_{\max} = H \ln(a_a n_0 H)$$

where the scale height  $H$  is given by

$$H = \frac{k_B T}{gm}$$

The pressure of the neutral atmosphere at the Jupiter surface is given by

$$p = n_0 k_B T \Rightarrow$$

$$n_0 = \frac{p}{k_B T} = \frac{100 \cdot 10^3}{1.38 \cdot 10^{-23} \cdot 300} = 2.4 \cdot 10^{25} \text{ m}^{-3}$$

For atomic hydrogen:

Here we use the mass of the atomic hydrogen in the expression for the scale height and the appropriate value for the absorption coefficient,  $\bar{\sigma}(H)$ .

$$H = \frac{k_B T}{gm} = \frac{1.38 \cdot 10^{-23} \cdot 300}{1.67 \cdot 10^{-27} \cdot 23.1} = 107 \text{ km}$$

$$z_{\max} = H \ln(a_a n_0 H) = 107 \cdot \ln(3 \cdot 10^{-18} \cdot 10^{-4} \cdot 2.4 \cdot 10^{-25} \cdot 107 \cdot 10^3) = 1700 \text{ km}$$

For molecular hydrogen:

Here we use the mass of the molecular hydrogen in the expression for the scale height and the appropriate value for the absorption coefficient  $\bar{\sigma}(H_2)$ .

$$H = \frac{k_B T}{gm} = \frac{1.38 \cdot 10^{-23} \cdot 300}{2 \cdot 1.67 \cdot 10^{-27} \cdot 23.1} = 54 \text{ km}$$

$$z_{\max} = H \ln(a_a n_0 H) = 54 \cdot \ln(6.1 \cdot 10^{-18} \cdot 10^{-4} \cdot 2.4 \cdot 10^{25} \cdot 54 \cdot 10^3) = 819 \text{ km}$$

Atomic hydrogen gives the more realistic altitude of the electron density peak.

b)

$$\frac{dn_e}{dt} = q - \alpha n_e^2$$

$$q = 0 \Rightarrow$$

$$\frac{dn_e}{dt} = -\alpha n_e^2 \Rightarrow$$

$$\int_{n_{e0}}^{n_e} \frac{dn_e}{n_e^2} = -\alpha \int_{t_0}^t dt \Rightarrow$$

$$\left[ -\frac{1}{n_e} \right]_{n_{e0}}^{n_e} = \alpha(t - t_0) \Rightarrow$$

$$\frac{1}{n_e} = \frac{1}{n_{e0}} + \alpha(t_0 - t) \Rightarrow$$

$$n_e = \frac{1}{\frac{1}{n_{e0}} + \alpha(t_0 - t)} = \frac{n_{e0}}{1 + \alpha n_{e0}(t_0 - t)}$$

Jupiter's rotation period is  $0.41 \cdot 24 \text{ h} = 35424 \text{ s}$ .

For atomic hydrogen:

$$n_{e0} = 2.5 \cdot 10^5 \text{ cm}^{-3}$$

$$\alpha = 10^{-12} \text{ cm}^3 \text{ s}^{-1}$$

Thus

$$n_e = \frac{2.5 \cdot 10^5}{1 + 10^{-12} \cdot 2.5 \cdot 10^5 \cdot \frac{35424}{2}} = 2.49 \cdot 10^5 \text{ cm}^{-3}$$

Virtually unchanged!

For molecular hydrogen:

$$n_{e0} = 2.5 \cdot 10^5 \text{ cm}^{-3}$$

$$\alpha = 10^{-8} \text{ cm}^3 \text{ s}^{-1}$$

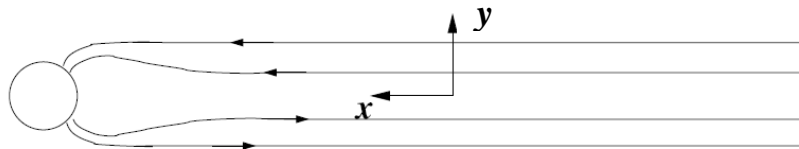
Thus

$$n_e = \frac{2.5 \cdot 10^5}{1 + 10^{-8} \cdot 2.5 \cdot 10^5 \cdot \frac{35424}{2}} = 5.5 \cdot 10^3 \text{ cm}^{-3}$$

Changes by a factor of 45.

2.

$$\mathbf{B} = \begin{cases} -B_0 \hat{\mathbf{x}} & , \quad y < -a \\ B_0 \hat{\mathbf{x}} \frac{3a^2 y - y^3}{2a^3} & , \quad -a \leq y \leq a \\ B_0 \hat{\mathbf{x}} & , \quad y > a \end{cases}$$



a)

$$\mathbf{j} = -\frac{1}{\mu_0} \frac{\partial B_x}{\partial y} \hat{\mathbf{z}} = \hat{\mathbf{z}} \begin{cases} 0 & , \quad y < -a \\ -\frac{1}{\mu_0} B_0 \frac{3a^2 - 3y^2}{2a^3} & , \quad -a \leq y \leq a \\ 0 & , \quad y > a \end{cases}$$

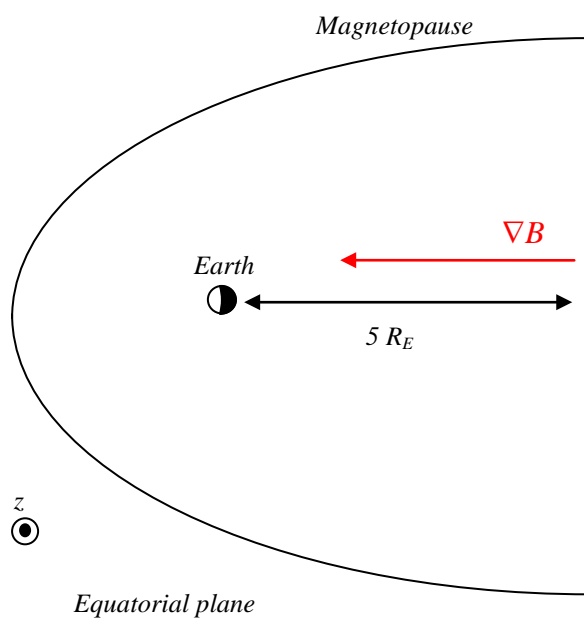
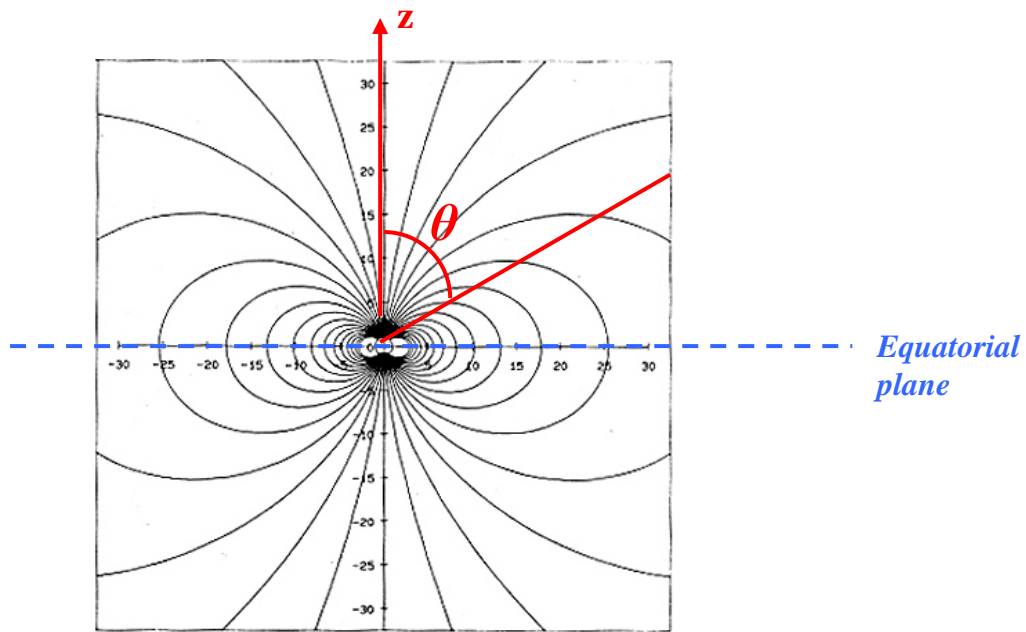
For  $y = 0$ :

$$j_z(x, 0) = -\frac{1}{\mu_0} B_0 \frac{3a^2}{2a^3} = -\frac{3}{2} \frac{B_0}{\mu_0 a} = -\frac{3}{2} \frac{10^{-8}}{4\pi \cdot 10^{-7} \cdot 2 \cdot 10^6} \text{ Am}^{-2} = 6.0 \cdot 10^{-9} \text{ Am}^{-2}$$

b)

$$I = l \int_{-a}^a |\mathbf{j}| dy = \frac{l}{\mu_0} \int_{-a}^a \left| \frac{\partial B_x}{\partial y} \right| dy = \frac{l}{\mu_0} |B_x(a) - B_x(-a)| = \frac{80 \cdot 6378 \cdot 10^3}{\mu_0} (20 \cdot 10^{-9}) = 8.1 \text{ MA}$$

3.



$$B^2 = B_p^2 R_E^6 r^{-6} \cos^2 \theta + \frac{B_p^2}{4} R_E^6 r^{-6} \sin^2 \theta$$

What is  $\nabla B$ ? It is actually easier to calculate  $\nabla B^2 = 2B\nabla B$ . In polar coordinates

$$\left(\nabla B^2\right)_r = \frac{\partial B^2}{\partial r} = B_p^2 R_E^6 \left(-6r^{-7} \cos^2 \theta - \frac{6}{4} r^{-7} \sin^2 \theta\right)$$

$$\theta = 90^\circ \Rightarrow$$

$$\left(\nabla B^2\right)_r = -\frac{3}{2} B_p^2 R_E^3 r^{-7}$$

The  $\theta$  component is zero:

$$\left(\nabla B^2\right)_\theta = \frac{1}{r} \frac{\partial B^2}{\partial \theta} = \frac{1}{r} B_p^2 R_E^6 r^{-6} (-2 \cos \theta \sin \theta + \frac{1}{4} 2 \cos \theta \sin \theta)$$

$$\theta = 90^\circ \Rightarrow$$

$$\left(\nabla B^2\right)_\theta = 0$$

Also

$$\theta = 90^\circ \Rightarrow$$

$$B = \frac{1}{2} B_p R_E^3 r^{-3}$$

Then

$$\nabla B = \hat{\mathbf{r}} \frac{1}{2B} \left(\nabla B^2\right)_r = -\frac{3}{2} \frac{B_p^2 R_E^6 r^{-7} 2}{2 B_p R_E^3 r^{-3}} \hat{\mathbf{r}} = -\frac{3}{2} B_p R_E^3 r^{-4} \hat{\mathbf{r}}$$

i.e. pointing towards Earth..

The drift velocity is

$$\mathbf{v}_{drift} = \frac{\mathbf{F} \times \mathbf{B}}{qB^2} = -\frac{\mu(\nabla B) \times \mathbf{B}}{qB^2}$$

Since all angles are 90 degrees

$$v_{drift} = \left| \mathbf{v}_{drift} \right| = \frac{\mu |\nabla B|}{qB}$$

Further

$$\mu = \frac{mv_{\perp}^2}{2B} = \frac{mv^2}{2B} = \frac{W}{B} \quad \text{since}$$

$$\alpha = 90^\circ \Rightarrow v_{\perp} = v$$

Thus

$$v_{drift} = \frac{W}{B} \frac{|\nabla B|}{qB} = \frac{W}{qB^2} |\nabla B| = \frac{W \cdot 3B_p R_E^3 \cdot 2^2 r^6}{q \cdot 2r^4 B_p^2 R_E^6} = \{r = 5 R_E\} = \frac{6 \cdot 25 \cdot W}{q B_p R_E} = 0.379 \frac{W}{q}$$

For electrons with  $W = 10^4 \text{ eV} = 10^4 q \text{ J}$  we get

$$v_{drift} = \frac{W}{B} \frac{|\nabla B|}{qB} = 0.379 \frac{10^4 q}{q} = 3790 \text{ m/s}$$

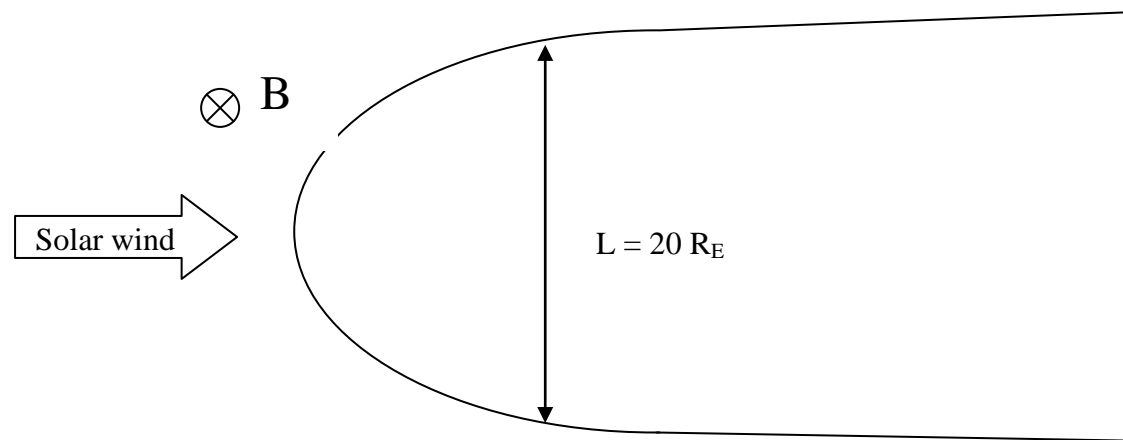
The revolution time  $T$  is

$$T = \frac{O}{v} = \frac{2\pi \cdot 5 R_E}{v} = 14.7 \text{ h}$$

For the ions  $T$  will be 16.8 years.

4.

Seeing the magnetosphere from "above":



The induced electric field from the solar wind is

$$\mathbf{E} = -\mathbf{v}_{SW} \times \mathbf{B}_{SW}$$

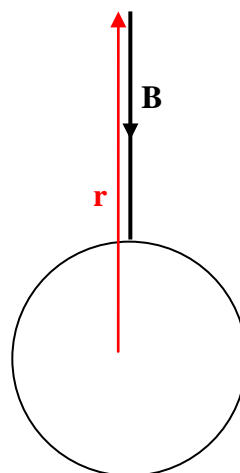
If we just care about the magnitude

$$E = vB$$

This gives a potential drop in the east-west direction over the magnetosphere of

$$EL = vBL = 350 \cdot 10^3 \cdot 7 \cdot 10^{-9} \cdot 20 \cdot 6378 \cdot 10^3 \approx 310 \text{ kV}$$

5.



The distance  $r$  from Earth's centre is  $10\,000 \text{ km} + 1 R_E = 16378 \text{ km}$ . With  $\theta = 0$  we get



$$B(r) = \frac{\mu_0 a}{2\pi} \frac{1}{r^3}$$

The electron is mirrored when the magnetic field is

$$B_{turn} = \frac{B}{\sin^2 \alpha} \Rightarrow$$

$$\frac{\mu_0 a}{2\pi} \frac{1}{r_{turn}^3} = \frac{\mu_0 a}{2\pi} \frac{1}{r^3 \sin^2 \alpha} \Rightarrow$$

$$r_{turn} = r(\sin \alpha)^{2/3} = 16378 \cdot (\sin 15^\circ)^{2/3} = 6655 \text{ km}$$

The altitude h will then be  $h = 6655 - 6378 = 273 \text{ km}$ . This means that this electron will have a reasonable chance to collide with a neutral atom or molecule to produce aurora.